

Algebraic Hopf invariants and rational models for mapping spaces

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Abstract

In this paper we will define an invariant $mc_\infty(f)$ of maps $f : X \rightarrow Y_{\mathbb{Q}}$ between a finite CW-complex and a rational space $Y_{\mathbb{Q}}$. We prove that this invariant is complete, i.e. $mc_\infty(f) = mc_\infty(g)$ if and only if f and g are homotopic. We will also construct an L_∞ -model for the based mapping space $Map_*(X, Y_{\mathbb{Q}})$ from a C_∞ -coalgebra and an L_∞ -algebra.

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1 Introduction

One of the most elementary questions in algebraic topology is: "Given two maps $f, g : X \rightarrow Y$, are f and g homotopic?" In general this is an extremely hard question but a lot of partial progress has been made. One example is the Hopf invariant, originally this was an invariant of maps $f : S^{4n-1} \rightarrow S^{2n}$ and can for example be used to show that the Hopf fibration is not null homotopic. The Hopf invariant has been generalized by many people, (see [17] and [5] and their references for more details about this), but one of the most recent generalizations is due to Sinha and Walter (see [17]). They generalize the Hopf invariant to maps $f : S^n \rightarrow Y_{\mathbb{Q}}$ between a sphere and a rational space $Y_{\mathbb{Q}}$ and prove that this generalization is a complete invariant, i.e. two maps f and g are homotopic if and only if their Hopf invariants coincide.

The main goal of this paper is to generalize the work of Sinha and Walter to maps between arbitrary spaces. We do this by defining a generalization of their Hopf invariant, which is a function $mc_{\infty} : Map_*(X, Y_{\mathbb{Q}}) \rightarrow \mathcal{MC}(X, Y)$ from the set of pointed maps between a finite simply connected CW complex X and a simply connected rational space $Y_{\mathbb{Q}}$, to the moduli space of Maurer-Cartan elements in a certain L_{∞} -algebra associated to X and Y . The main Theorem of this paper is given by:

Theorem 1.1. Let $f, g : X \rightarrow Y$ be two maps between a finite simply connected CW-complex X and a simply connected space Y , then f and g are homotopic in the model category of rational spaces if and only if $mc_{\infty}(f) = mc_{\infty}(g)$.

To prove this theorem we will first generalize the statement of the theorem and define a complete invariant of homotopy classes of maps between coalgebras over a cooperad \mathcal{C} . To do this it will be necessary to define an L_{∞} -algebra structure on the convolution algebra between a \mathcal{C} -coalgebra and a $\Omega_{op}\mathcal{C}$ -algebra, where $\Omega_{op}\mathcal{C}$ is the operadic cobar construction on the cooperad \mathcal{C} . We will do this in a more general setting and define an L_{∞} -structure on the convolution algebra relative to an operadic twisting morphism in the following theorem.

This L_{∞} -structure generalizes a construction from a paper by Dolgushev, Hoffnung and Rogers [8] and a construction from the book by Loday and Vallette [15]. In [8] they construct this L_{∞} -structure for the canonical twisting morphism $\iota : \mathcal{C} \rightarrow \Omega_{op}\mathcal{C}$ and in [15] they define a Lie algebra relative to a binary quadratic twisting morphism. We will also give a shorter and more conceptual alternative proof for this theorem.

Theorem 1.2. Let $\tau : \mathcal{C} \rightarrow \mathcal{P}$ be an operadic twisting morphism and let C be a \mathcal{C} algebra and L a \mathcal{P} -algebra, then there exists an L_{∞} -structure on the convolution algebra $Hom_{\mathbb{K}}(C, L)$.

As a consequence of this theorem we will also find a way to generalize a theorem by Berglund from [4] about models for mapping spaces.

Theorem 1.3. Let C be a finite dimensional C_{∞} -coalgebra model for a finite simply connected CW-complex X and L an L_{∞} -model for a simply connected rational space $Y_{\mathbb{Q}}$ of finite \mathbb{Q} -type, then there exists an explicit L_{∞} -structure on the space $Hom_{\mathbb{K}}(C, L)$ such that $Hom_{\mathbb{K}}(C, L)$ becomes a model for $Map_*(X, Y_{\mathbb{Q}})$. In particular,

1. There is a bijection between $[X, Y_{\mathbb{Q}}] \simeq \mathcal{MC}(Hom_{\mathbb{K}}(C, L))$.
2. For every Maurer-Cartan element τ we have an isomorphism

$$\pi_n(Map_*(X, Y_{\mathbb{Q}}), \tau) \otimes \mathbb{Q} \simeq H_n(Hom_{\mathbb{K}}(C, L)^{\tau}).$$

Here the morphism corresponding to τ is the base point for the homotopy groups π_n and $Hom_{\mathbb{K}}(C, L)^\tau$ is the L_∞ -algebra $Hom_{\mathbb{K}}(C, L)$ twisted by τ .

In the paper by Berglund this theorem assumes that C is strictly coassociative cocommutative coalgebra, we improve this theorem by relaxing the assumption that C has to be coassociative. In our construction C can also be a C_∞ -coalgebra which is a cocommutative coalgebra which is only coassociative up to homotopy.

As a corollary of this theorem we will find an alternative proof of a theorem by Buijs and Gutiérrez which states that $Hom_{\mathbb{K}}(\tilde{H}_*(X; \mathbb{Q}), \pi_*(Y) \otimes \mathbb{Q})$ can be equipped with an L_∞ -structure such that it becomes a model for the mapping space $Map_*(X, Y)$.

Corollary 1.1. Under the assumptions of Theorem 1.3 there exists an explicit L_∞ -structure on the space $Hom_{\mathbb{K}}(\tilde{H}_*(X; \mathbb{Q}), \pi_*(Y) \otimes \mathbb{Q})$ such that this becomes a model for the mapping space as in Theorem 1.3.

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2 Conventions

In this paper we will follow the following conventions.

Convention 2.1. Throughout this paper we will assume that \mathbb{K} is a field of characteristic 0, and in the topological part of the paper we will work over the rationals, furthermore we will assume that all homology, cohomology, homotopy and cohomotopy groups are taken with rational coefficients. For simplicity we will often omit this from the notation.

Convention 2.2. We assume that all the operads and cooperads we consider are connected, i.e. $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbb{K}$ for operads and $\mathcal{C}(0) = 0$ and $\mathcal{C}(1) = \mathbb{K}$ for cooperads.

Convention 2.3. We will assume that all cooperads and coalgebras we will consider are conilpotent. This means that the coradical filtration is exhaustive. See Section 5.8.4 and 5.8.5 in [15] for a definition and more details.

Convention 2.4. In this paper we will tacitly assume that all spaces are based and all the mapping spaces are spaces of pointed maps. We will also assume that all spaces except for the mapping spaces are 1-reduced, i.e. have one zero-cell and no one-cells. In particular all the spaces, except for the mapping spaces, are simply connected. Since we are studying based mapping spaces we will always work with reduced homology and cohomology, so the notation $\tilde{H}_*(X)$ will mean the reduced homology of the space X with coefficients in \mathbb{Q} . As a consequence of this all algebras and coalgebras will be non-(co)unital.

We will also assume that all homotopy groups are rational homotopy groups, to ease the notation we will write $\pi_*(X)$ instead of $\pi_*(X) \otimes \mathbb{Q}$ to denote the rational homotopy groups of a space X .

Definition 2.1. Let X and Y be based spaces, then we denote the space of based maps between X and Y by $Map_*(X, Y)$.

Convention 2.5. In this we will mainly work with \mathbb{Z} -graded chain complexes, which we will grade homologically, so the differential will always have degree -1 . The only exception to this convention is the cohomology of a space which we will grade cohomologically.

Definition 2.2. The rational homotopy Lie algebra of a space X is denoted by $\pi_*(X)$ and the linear dual of the rational homotopy groups is denoted by $\pi^*(X)$ and will be called the cohomotopy groups.

Remark 2.1. Note that our definition of the cohomotopy groups is different from the standard notion of cohomotopy groups. The cohomotopy group $\pi^n(X)$ is not the space of maps $[X, S^n]$.

Part I

Preliminaries

3 Twisting morphisms, Koszul duality and bar constructions

In this section we will recall the most important concepts about operads, cooperads, twisting morphisms and the bar and cobar constructions.

3.1 Convolution algebras and twisting morphisms

In this paper we will use several different bar constructions, in this section we will briefly recall the definitions and introduce some notation. All our conventions and definitions are based on the book [15], unless stated otherwise.

Definition 3.1. The operadic bar construction on an operad \mathcal{P} is denoted by $B_{op}\mathcal{P}$ and the operadic cobar construction on a cooperad \mathcal{C} is denoted by $\Omega_{op}\mathcal{C}$.

Definition 3.2. Let $\alpha : \mathcal{C} \rightarrow \mathcal{P}$ be an operadic twisting morphism, (C, Δ_C, d_C) a \mathcal{C} -coalgebra and (A, μ_A, d_A) a \mathcal{P} -algebra. The convolution algebra $Hom_{\mathbb{K}}(C, A)$ is the dg vector space of all linear maps from C to A . The differential is defined by

$$\partial(f) = d_A \circ f - (-1)^{|f|} f \circ d_C.$$

The Maurer-Cartan operator $\star_\alpha : Hom_{\mathbb{K}}(C, A) \rightarrow Hom_{\mathbb{K}}(C, A)$ is defined by

$$\star_\alpha(f) = C \xrightarrow{\Delta_C} \mathcal{C} \circ C \xrightarrow{\alpha \circ f} \mathcal{P} \circ A \xrightarrow{\mu_A} A.$$

Definition 3.3. A twisting morphism relative to α is defined as a linear map $\tau : C \rightarrow A$ such that τ satisfies the Maurer-Cartan equation, which is given by

$$\partial(\tau) + \star_\alpha(\tau) = 0.$$

The set of Maurer-Cartan elements is denoted by $MC(C, A)$.

3.2 Bar and cobar constructions for algebras over an operad

The main reason we care about twisting morphisms in this paper is because they are represented by the bar and cobar constructions. The bar construction will provide us with functorial fibrant replacements in the model category of \mathcal{C} -coalgebras and the cobar construction will provide us with functorial cofibrant replacements in the category of \mathcal{P} -algebras. The bar and cobar constructions also give us a way to relate the rational homology and rational homotopy groups to each other (see for example [3]). In this section we will define the bar and cobar construction for algebras and coalgebras relative to an operadic twisting morphism $\alpha : \mathcal{C} \rightarrow \mathcal{P}$, between a cooperad \mathcal{C} and an operad \mathcal{P} . The bar and cobar construction form an adjoint pair of functors between the category of \mathcal{C} -coalgebras and \mathcal{P} -algebras:

$$\Omega_\alpha : \{\text{conilpotent } \mathcal{C}\text{-coalgebras}\} \rightleftarrows \{\mathcal{P}\text{-algebras}\} : B_\alpha.$$

The constructions are given in the following definitions.

Definition 3.4. Let A be a \mathcal{P} -algebra, then we define the bar construction $B_\alpha A$ on A as $(B_\alpha A = \mathcal{C}(A), d_B)$. Where $\mathcal{C}(A)$ is the cofree \mathcal{C} -coalgebra on the underlying vector space of A and $d_B = d_1 + d_2$. The differential d_1 is the unique extension of d_A to $\mathcal{C}(A)$ and d_2 is the unique coderivation extending the following map

$$\mathcal{C} \circ A \xrightarrow{\alpha \circ Id_A} \mathcal{P} \circ A \xrightarrow{\mu_A} A.$$

Similarly we also have the cobar construction which is defined as follows.

Definition 3.5. Let C be a \mathcal{C} -coalgebra, then we define the cobar construction $\Omega_\alpha C = (\mathcal{P}(C), d_\Omega)$ as the free \mathcal{P} -algebra on the underlying vector space of C . The differential $d_\Omega = d_1 + d_2$, where d_1 is the unique extension of d_C to $\mathcal{P}(C)$ and d_2 is the unique derivation that extends the following map

$$C \xrightarrow{\Delta_C} \mathcal{C} \circ C \xrightarrow{\alpha \circ Id_C} \mathcal{P} \circ C.$$

The following theorem states that the set of Maurer-Cartan elements are represented by the bar and cobar construction.

Theorem 3.1. Let $\alpha : \mathcal{C} \rightarrow \mathcal{P}$ be an operadic twisting morphism, C a \mathcal{C} -coalgebra and A a \mathcal{P} -algebra. Then there are natural bijections

$$Hom_{\mathcal{P}\text{-alg}}(\Omega_\alpha C, A) \cong MC_\alpha(C, A) \cong Hom_{\mathcal{C}\text{-coalg}}(C, B_\alpha A).$$

For the proof see Proposition 11.3.1 in [15].

3.3 Universal twisting morphisms and Koszul twisting morphisms

The bar and cobar construction have the universal property that every twisting morphism factors uniquely through two universal twisting morphisms associated to the bar and cobar construction. For more details see Section 6.5.11 in [15].

Proposition 3.1. Let $\phi : C \rightarrow A$ be a twisting morphism relative to an operadic twisting morphism $\alpha : \mathcal{C} \rightarrow \mathcal{P}$. Then there exist universal twisting morphisms $\pi : B_\alpha A \rightarrow A$ given by the projection of $B_\alpha A$ on A and $\iota : C \rightarrow \Omega_\alpha C$ given by the inclusion of C into $\Omega_\alpha C$. Such that ϕ factors uniquely through π in the sense that there is a unique map of \mathcal{C} -coalgebras $f_\phi : C \rightarrow B_\alpha A$ such that $\phi = \pi \circ f_\phi$. Similarly there is a unique map of \mathcal{P} -algebras $g_\phi : \Omega_\alpha C \rightarrow A$ such that $\phi = g_\phi \circ \iota$. This is summarized in the following diagram

$$\begin{array}{ccc}
 & B_\alpha A & \\
 f_\phi \nearrow & & \searrow \pi \\
 C & \xrightarrow{\phi} & A \\
 \iota \searrow & & \nearrow g_\phi \\
 & \Omega_\alpha C &
 \end{array}$$

The main reason to consider the bar and cobar construction is because they give functorial fibrant and cofibrant replacements, this will be shown in Section 5, the following theorem states that the bar-cobar resolution and the cobar-bar resolution are in fact resolutions of a coalgebra and an algebra.

Theorem 3.2. The counit $\epsilon_\alpha : \Omega_\alpha B_\alpha A \rightarrow A$ and unit $\nu_\alpha : C \rightarrow B_\alpha \Omega_\alpha C$ of the bar-cobar adjunction are weak equivalences.

4 The L_∞ operad and L_∞ -algebras

One of the most important operads in this paper will be the L_∞ -operad. The L_∞ -operad is the operad describing Lie algebra up to homotopy and is heavily used in for example deformation theory and rational homotopy theory. In this section we will recall most of the basics of the theory of L_∞ -algebras, for more details see for example [13], [7] and [4].

Remark 4.1. In this paper we will use some unusual conventions about the grading for the L_∞ -operad. We will assume that every generating operation in the L_∞ -operad has degree -1 instead of the usual convention where the generating operation in arity n has degree $2 - n$. The reason we use this convention is because it seems to be more natural in the applications we want to consider. The reader who wants to translate everything to the usual conventions should just (de)suspend everything.

Definition 4.1. The cocommutative cooperad \mathcal{COCOM} is the cooperad given by the symmetric sequence $\mathcal{COCOM}(k) = \mathbb{K}\mu_k$ which is one dimensional in each arity and with the trivial symmetric group action. The decomposition map is given by

$$\Delta_{\mathcal{COCOM}}(\mu_n) = \sum_{p=1}^n \sum_{i=1}^p \mu_p \circ_i \mu_{n-p+1}.$$

Definition 4.2. The L_∞ -operad is the operad given by the cobar construction on the cooperad \mathcal{COCOM} . An L_∞ -algebra L is an algebra over the L_∞ -operad, more specifically it is a graded vector space L with for each $n \geq 1$ an operation $l_n : L^{\otimes n} \rightarrow L$ of degree -1 , such that the operations l_n are graded symmetric and satisfy a shifted version of the higher Jacobi identities.

Definition 4.3. Let (A, μ) be a commutative algebra and (L, l_1, l_2, \dots) an L_∞ -algebra, then we define the extension of scalars of L by A as the L_∞ -algebra whose underlying vector spaces is $A \otimes L$, with the operations given by $l_n(a_1 \otimes x_1, \dots, a_n \otimes x_n) = a_1 \cdot a_2 \cdot \dots \cdot a_{n-1} \cdot a_n \otimes l_n(x_1, \dots, x_n)$.

To each L_∞ -algebra we can associate a simplicial set as follows, see [13] for more details.

Definition 4.4. Let L be an L_∞ -algebra, a Maurer-Cartan element $x \in L$ is a degree 0 element satisfying the Maurer-Cartan equation which is given by

$$\sum_{n \geq 1} \frac{1}{n!} l_n(\tau, \dots, \tau) = 0.$$

The set of Maurer-Cartan elements in an L_∞ -algebra is denoted by $MC(L)$. To each nilpotent L_∞ -algebra L we associate a simplicial set $MC_\bullet(L)$ whose set of n -simplices is given by $MC(L \otimes \Omega_n)$, where Ω_n is the commutative algebra of polynomial de Rham forms on the n -simplex. The face and degeneracy maps are the ones induced by the face and degeneracy maps of Ω_\bullet .

The following theorem is comes from [13].

Theorem 4.1. The simplicial set $MC_\bullet(L)$ is a Kan complex.

Using this definition we can now define rational models for a space X .

Definition 4.5. Let X be a simply connected space of finite \mathbb{Q} -type, a rational L_∞ -model L for the space X is an L_∞ -algebra L such that $MC_\bullet(L)$ is rationally equivalent to the space X .

In [13] it is shown that L_∞ -models rational models for simply connected spaces of finite \mathbb{Q} -type exist.

In this paper we are mainly interested in the set of path components of the simplicial set $MC_\bullet(L)$. Since the set of zero simplices of $MC_\bullet(L)$ is given by $MC(L)$, the Maurer-Cartan elements of L , being in the same path component induces an equivalence relation on $MC(L)$. This equivalence relation will be called homotopy or gauge equivalence. In particular two Maurer-Cartan elements $x, y \in L$ are gauge equivalent if there exists a Maurer-Cartan element $z \in L \otimes \Omega_1$ such that z is of the form $z = \sum z_i \otimes P_i(t) + z'_i \otimes Q_i(t)dt$, where the sum runs over a basis $\{z_i\}$ of L and P and Q are polynomials in t . The elements x and y are then called gauge equivalent if $\sum z_i \otimes P_i(0) = x$ and $\sum z_i \otimes P_i(1) = y$.

Definition 4.6. The moduli space of Maurer-Cartan elements in an L_∞ -algebra L is the set of Maurer-Cartan elements modulo the relation of gauge equivalence. The moduli space of Maurer-Cartan elements is denoted by $\mathcal{MC}(L)$.

Remark 4.2. In deformation theory it is common to have an alternative definition of gauge equivalence given by the action of the "Lie group" associated to the L_∞ algebra on the set of Maurer-Cartan elements, in the papers [7] and [9] it is proven that gauge and homotopy equivalence are the same equivalence relation. Since the word homotopy equivalence is already quite overused in this paper we shall refer to this equivalence relation as gauge equivalence.

4.1 C_∞ -models for spaces

In the previous section we have described the L_∞ approach to rational homotopy theory, but there is also a second approach with C_∞ -coalgebras.

In this section we will recall the basic facts about this approach. We start by defining the C_∞ -cooperad, this is a fibrant replacement of the cocommutative cooperad.

Definition 4.7. The C_∞ -cooperad is the operad defined as the bar construction on the operad \mathcal{LIE} , i.e. $C_\infty = B_{op}\mathcal{LIE}$.

The next step is to use the following theorem by Quillen, which is the main theorem of [16].

Theorem 4.2. There exists a functor $\mathcal{C}\lambda : Top_{*,1} \rightarrow CDGC$ which induces an equivalence of homotopy categories between 1-reduced rational spaces of finite \mathbb{Q} -type to cocommutative differential graded coalgebras.

Using this theorem we define a C_∞ -model for a space as follows.

Definition 4.8. Let X be a simply connected space of finite \mathbb{Q} -type, a C_∞ -coalgebra C is a C_∞ -model for X if there exists a zig-zag of quasi-isomorphisms between $\mathcal{C}\lambda(X)$ and C .

4.2 The Pre-Lie algebra associated to an operad

The goal of this section is to recall how we can associate a Pre-Lie algebra to an operad. See [15] for more details.

Proposition 4.1. Let \mathcal{P} be an operad, then there exists a Pre-Lie algebra structure on the space $\bigoplus_{n \geq 1} \mathcal{P}(n)$ with the Pre-Lie operation given by

$$\{\mu, \nu\} = \sum_{i=1}^n \sum_P (\mu \circ_i \nu)^{\sigma_P},$$

with $\mu \in \mathcal{P}(n)$ and $\nu \in \mathcal{P}(m)$ and the second sum runs over all ordered partitions $P \in Ord(1, \dots, 1, n-i+1, 1, \dots, 1)$, with $n-i+1$ on the i th spot.

4.3 The convolution operad

In this section we will recall the definition of the convolution operad. This is an operad associated to the space $Hom(\mathcal{C}, \mathcal{P})$ of linear maps between a cooperad \mathcal{C} and an operad \mathcal{P} . The convolution operad was first defined in [1]. For more details see also [15] on which this section is based.

Definition 4.9. Let \mathcal{C} be a cooperad and \mathcal{P} be an operad, then we define a operad structure on the space of linear maps $Hom(\mathcal{C}, \mathcal{P})$ as follows.

The arity n part of the convolution operad is defined as the space $Hom_{\mathbb{K}}(\mathcal{C}(n), \mathcal{P}(n))$. The symmetric group action on $Hom_{\mathbb{K}}(\mathcal{C}(n), \mathcal{P}(n))$ is defined by

$$f^\sigma(x) = \sigma(f(x^{\sigma^{-1}})).$$

It turns out that the cocommutative cooperad plays a special role, since it will act as a "unit" with respect to the convolution operad. The following lemma will be important in section 7.

Lemma 4.1. The convolution operad between the cocommutative cooperad $COCOM$ and an operad \mathcal{P} is isomorphic to \mathcal{P} .

Proof. We have to show that there is an isomorphism between $Hom_{\mathbb{K}}(\mathcal{COCOM}, \mathcal{P})$ and \mathcal{P} . To do this we define an explicit isomorphism

$$\phi : Hom_{\mathbb{K}}(\mathcal{COCOM}, \mathcal{P}) \rightarrow \mathcal{P},$$

which is given by sending a map $f : \mathcal{COCOM} \rightarrow \mathcal{P}$ to its image in \mathcal{P} , i.e. in arity n component of this map is given by

$$\phi(f) = f(\mu_n).$$

Where μ_n is the basis element of $\mathcal{COCOM}(n)$. Since \mathcal{COCOM} is one dimensional in each arity, the map ϕ is an aritywise isomorphism of vector spaces. The morphism ϕ commutes with the symmetric group action since the symmetric group action on \mathcal{COCOM} is trivial. Therefore the action on a map $f \in Hom_{\mathbb{K}}(\mathcal{COCOM}, \mathcal{P})$ is given by $\sigma(f)$, which is the same as the action coming from \mathcal{P} . It is a straightforward but tedious check that the morphism ϕ commutes with the operadic composition maps. \square

5 Model structures on algebras and coalgebras

In this paper we will use several different model categories, in this section we will introduce the model categories we will use. For an introduction to the theory of model categories we recommend [11] from which we have taken most of the definitions and conventions.

First we will define a model structure on \mathcal{P} algebras which will induce a model structure on \mathcal{C} coalgebras. This was originally done by Hinich in [14].

Theorem 5.1. Let \mathcal{P} be an operad, then a model structure on the category of \mathcal{P} algebras is given by

- The weak equivalences are given by the quasi-isomorphisms.
- The fibrations are given by maps that are degree wise surjective.
- The cofibrations are the maps with the left lifting property with respect to acyclic fibrations.

The following theorem is Theorem 3.11 in [10] and provides us with a model structure on the category of conilpotent coalgebras over a cooperad \mathcal{C} .

Theorem 5.2. Let \mathcal{C} be a dg cooperad and $\tau : \mathcal{C} \rightarrow \mathcal{P}$ be an operadic twisting morphism to a dg operad \mathcal{P} . Then there is a model structure on the category of conilpotent \mathcal{C} coalgebras such that,

- The weak equivalences are given by the maps $f : C \rightarrow D$ such that $\Omega_{\tau} f : \Omega_{\tau} C \rightarrow \Omega_{\tau} D$ is a quasi isomorphism of \mathcal{P} algebras.
- The cofibrations are the morphisms $f : C \rightarrow D$ such that f is a degree wise monomorphism.
- The fibrations are the morphisms with the right lifting property with respect to the acyclic cofibrations.

The following theorem is Proposition 3.15 from [10].

Theorem 5.3. Let $\tau : \mathcal{C} \rightarrow \mathcal{P}$ be an operadic twisting morphism, assume that \mathcal{C} and \mathcal{P} are weight graded and that τ respects the weight grading. In this case there is a Quillen equivalence between the categories of \mathcal{P} -algebras and \mathcal{C} -coalgebras if and only if the twisting morphism τ is Koszul.

In the following proposition we will describe the fibrant and cofibrant objects in the model categories of algebras and coalgebras.

Proposition 5.1. In the model category of algebras over an operad \mathcal{P} every object is fibrant, the cofibrant objects are retracts of quasi-free algebras $(\mathcal{P}(V), d)$ equipped with an exhaustive filtration on V , i.e. there is a filtration of the form

$$V_0 = \{0\} \subseteq V_1 \subseteq \dots \subseteq \text{Colim}_i V_i = V,$$

such that $d(V_i) \subseteq \mathcal{P}(V_{i-1})$. In the model category of coalgebras over a cooperad \mathcal{C} every object is cofibrant, the fibrant objects are given by the quasi-free \mathcal{C} -coalgebras.

Proof. For the proof of the algebra case see [14] and for the proof of the coalgebra case see Theorem 2.1 in [18]. \square

Remark 5.1. From Proposition 5.1 it follows that in the algebra case, $\Omega_\tau C$ the cobar construction on a \mathcal{C} -coalgebra C with respect to a Koszul twisting morphism $\tau : \mathcal{C} \rightarrow \mathcal{P}$ is a cofibrant object. Similarly in the coalgebra case, all coalgebras of the form $B_\tau A$, for some \mathcal{P} -algebra A are fibrant. Therefore we can define functorial cofibrant replacements by applying the cobar bar resolution in the algebra case. In the coalgebra case we have a functorial fibrant replacement given by the bar cobar resolution.

The following two lemmas will be important in section 10 to prove the completeness of the Hopf invariants. The first lemma is Lemma 4.9 in [11] and the second lemma is Ken Brown's Lemma and can be found as Lemma 9.9 in [11].

Lemma 5.1. Let A be a cofibrant object in a model category \mathbf{C} and let $p : Y \rightarrow X$ be an acyclic fibration. Then composition with p induces a bijection

$$p_* : \pi^l(A, Y) \rightarrow \pi^l(A, X),$$

where $\pi^l(A, Y)$ is the set of left homotopy classes of maps between A and Y .

Lemma 5.2. Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a functor between two model categories \mathbf{C} and \mathbf{D} , such that F carries acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.

6 The Homotopy transfer theorem, \mathcal{P}_∞ algebras and \mathcal{C}_∞ coalgebras

The final ingredient we need to define the Hopf invariants is the homotopy transfer theorem and the notion of algebras and coalgebras up to homotopy. In this paper we will use slightly different definitions for \mathcal{P}_∞ algebras than in the book by Loday and Vallette [15], instead of defining a \mathcal{P}_∞ algebra as an algebra over the Koszul resolution $\Omega_{op} \mathcal{P}^i$, we will define a \mathcal{P}_∞ algebra as an algebra over the bar cobar resolution $\Omega_{op} B_{op} \mathcal{P}$. This has the advantage that this resolution always exists and does not

require the operad \mathcal{P} to be Koszul. Because the category of $\Omega_{op}\mathcal{P}^i$ and the category of $\Omega_{op}B_{op}\mathcal{P}$ are Quillen equivalent (see Theorem 12.5A in [12]), it does not matter from a homotopy theoretical perspective if we would work with $\Omega_{op}\mathcal{P}^i$ algebras or with $\Omega_{op}B_{op}\mathcal{P}$ algebras. The main disadvantage is that all the formulas will become more complicated than necessary, but since we will mainly use the homotopy transfer theorem as a theoretical tool to prove completeness of the algebraic Hopf invariant, this is not a serious disadvantage. Although the results are presented slightly different than in [15], all the proofs and details can be found in [15] or are completely analogous to the proofs and details there. For other papers on the homotopy transfer theorem and homotopy algebras see also [2] and [8].

There are several equivalent definitions for \mathcal{P}_∞ algebras.

Definition 6.1. A \mathcal{P}_∞ -algebra is an algebra over $\Omega_{op}B_{op}\mathcal{P}$, the cobar-bar resolution of \mathcal{P} .

The following theorem gives three alternative ways of describing \mathcal{P}_∞ -structures on a graded vector space A , it can be found in [15] as Theorem 10.1.22.

Theorem 6.1. A \mathcal{P}_∞ -algebra structure on a dg vector space A can be described in the following ways:

1. By a morphism of operads $f : \Omega_{op}B_{op}\mathcal{P} \rightarrow \text{End}_A$, where End_A is the endomorphism operad of the dg vector space A .
2. By an operadic twisting morphism $\tau : B_{op}\mathcal{P} \rightarrow \text{End}_A$.
3. By a morphism of cooperads $g : B_{op}\mathcal{P} \rightarrow B_{op}\text{End}_A$
4. Or by a square zero coderivation on $B_{op}\mathcal{P}(A)$, the free $B_{op}\mathcal{P}$ coalgebra cogenerated by A .

Therefore we have bijections between the following sets which all describe the set of \mathcal{P}_∞ structures,

$$\text{Hom}_{\text{Operads}}(\Omega_{op}B_{op}\mathcal{P}, \text{End}_A) \cong \text{Tw}(B\mathcal{P}, \text{End}_A) \cong$$

$$\text{Hom}_{\text{Cooperads}}(B_{op}\mathcal{P}, B_{op}\text{End}_A) \cong \text{Codi}f(B_{op}\mathcal{P}(A)).$$

The category of \mathcal{P}_∞ algebras can be equipped with two types of morphisms. The first type of morphisms are morphisms as $\Omega_{op}B_{op}\mathcal{P}$ algebras that commute with all the operations coming from the operad $\Omega_{op}B_{op}\mathcal{P}$, these morphisms will be called strict morphisms of \mathcal{P}_∞ -algebras. The second type of morphisms are the morphisms that commute only up to a sequence of coherent homotopies with the operations coming from the operad $\Omega_{op}B_{op}\mathcal{P}$. These morphism are called ∞ -morphisms and are defined in the following definitions. The advantage of ∞ -morphisms is that ∞ -quasi-isomorphisms are always invertible and that the category of \mathcal{P}_∞ -algebras with ∞ -morphisms is equivalent to the homotopy category of \mathcal{P} -algebras.

Definition 6.2. An ∞ -morphism $f : A \rightarrow B$ between two \mathcal{P}_∞ algebras A and B is a morphism $f : B_{op}\mathcal{P}(A) \rightarrow B_{op}\mathcal{P}(B)$ of $B_{op}\mathcal{P}$ -coalgebras. This is equivalent to a sequence of maps $f_i : A^{\otimes i'} \rightarrow B$, where i runs over a basis of $B_{op}\mathcal{P}$ and i' is the arity of the basis element indexed by i , satisfying certain coherence conditions. An ∞ -morphism $f : A \rightarrow B$ is called an ∞ -quasi-isomorphism if the component $f_1 : A \rightarrow B$ is a quasi-isomorphism, where f_1 is the morphism corresponding to the operadic unit.

The main reason we are using \mathcal{P}_∞ algebras in this paper is because of the homotopy transfer theorem.

Theorem 6.2. Suppose that we have a homotopy retract of vector spaces (V, d_V) of (W, d_W) , i.e. we have maps

$$\begin{array}{ccc} & & \\ & \begin{array}{c} \text{---} p \text{---} \\ \text{---} i \text{---} \end{array} & \\ i \circ & W & \longrightarrow V. \end{array}$$

Such that the maps i and p are quasi-isomorphisms of vector spaces and

$$Id_W - ip = d_W h + h d_W.$$

Suppose that we have a \mathcal{P} -structure on W , then there exists a \mathcal{P}_∞ -structure on V and an ∞ -morphism i' such that W and V are ∞ -quasi-isomorphic and the map i' is an ∞ -quasi-isomorphism such that $i'_{(1)}$ is equal to i .

Dually we can also define the notion of a coalgebra up to homotopy. This is done completely analogous to the algebra case

Definition 6.3. A \mathcal{C}_∞ -coalgebra is a coalgebra over the cooperad $B_{op}\Omega_{op}\mathcal{C}$ and an ∞ -morphism $f : C \rightarrow D$ of \mathcal{C}_∞ -coalgebras is a morphism of $\Omega_{op}\mathcal{C}$ -algebras $\hat{f} : \Omega_\pi C \rightarrow \Omega_\pi D$. This is equivalent to a sequence of maps $f_i : C \rightarrow D^{\otimes i'}$, where i runs over a basis of $\Omega_{op}\mathcal{C}$ and i' is the arity of the element indexed by i , satisfying a sequence of compatibility conditions.

Similarly to coalgebras we also have homotopy transfer theorem for \mathcal{C}_∞ -coalgebras.

Theorem 6.3. Let (V, d_V) be a retract of (W, d_W) as in Theorem 6.2 and suppose that we have \mathcal{C} -coalgebra structure on W . Then there exists a \mathcal{C}_∞ -structure on V such that V and W are ∞ -quasi-isomorphic and the morphism i extends to an ∞ -quasi-isomorphism $i' : W \rightarrow V$.

This theorem is not explicitly stated or proven in [15], but the proof is completely analogous to the algebra case.

Because of the homotopy transfer theorems it is always possible to equip the homology of a \mathcal{P} -algebra A (\mathcal{C} -coalgebra C) with an ∞ -structure such that we have a weak equivalence between A and $H_*(A)$ (C and $H_*(C)$). From now on we will always assume that the homology is equipped with the appropriate ∞ -structure.

Convention 6.1. In the rest of this paper we will assume that whenever we take the homology of a \mathcal{P} -algebra A (\mathcal{C} -coalgebra C) it is equipped with a \mathcal{P}_∞ -structure (\mathcal{C}_∞ -structure), such that A and $H_*(A)$ (C and $H_*(C)$) are quasi-isomorphic.

Part II

Algebraic Hopf invariants

7 An L_∞ -structure on $\text{Hom}(C, A)$

Let C be a \mathcal{C} -coalgebra and A a \mathcal{P} -algebra and let $\tau : C \rightarrow \mathcal{P}$ be an operadic twisting morphism. In this section we will describe an L_∞ -structure on the convolution algebra $\text{Hom}_{\mathbb{K}}(C, A)$, such that the Maurer-Cartan elements of this L_∞ -algebra are the twisting morphisms relative to an operadic twisting morphism $\tau : C \rightarrow \mathcal{P}$. We will also show that

the gauge equivalence relation on the Maurer-Cartan elements will be equivalent to the homotopy relation on the corresponding morphisms of \mathcal{P} -algebras.

To construct an L_∞ -structure on $Hom_{\mathbb{K}}(C, A)$ we first prove in Proposition 7.1 that $Hom_{\mathbb{K}}(C, A)$ is an algebra over the convolution operad $Hom_{\mathbb{K}}(\mathcal{C}, \mathcal{P})$. To construct the L_∞ -structure on $Hom_{\mathbb{K}}(C, A)$ we then need to define a morphism from L_∞ to $Hom_{\mathbb{K}}(\mathcal{C}, \mathcal{P})$.

Since the L_∞ -operad is defined as the cobar construction on \mathcal{COCOM} specifying a morphism from L_∞ to an operad \mathcal{Q} is the same as giving a twisting morphism $\mathcal{COCOM} \rightarrow \mathcal{Q}$. Because of Lemma 4.1 there is an isomorphism between $Hom_{\mathbb{K}}(\mathcal{COCOM}, \mathcal{Q})$ and \mathcal{Q} . A twisting morphism from \mathcal{COCOM} to \mathcal{P} is therefore the same as a Maurer-Cartan element in the pre-Lie algebra associated to \mathcal{Q} .

A twisting morphism $\tau : \mathcal{C} \rightarrow \mathcal{P}$ is therefore equivalent to a map of operads $L_\infty \rightarrow Hom_{\mathbb{K}}(\mathcal{C}, \mathcal{P})$ and therefore defines an L_∞ -structure on $Hom_{\mathbb{K}}(C, A)$.

We begin by showing how the convolution algebra is an algebra over the convolution operad.

Proposition 7.1. Let (C, Δ_C) be a \mathcal{C} -coalgebra and (A, μ_A) a \mathcal{P} -algebra, the convolution algebra $Hom_{\mathbb{K}}(C, A)$ is then an algebra over the convolution operad $Hom_{\mathbb{K}}(\mathcal{C}, \mathcal{P})$.

Proof. The algebra structure on $Hom_{\mathbb{K}}(C, A)$ is defined as follows. Let $\gamma \in Hom_{\mathbb{K}}(\mathcal{C}(n), \mathcal{P}(n))$ and $f_1, \dots, f_n \in Hom_{\mathbb{K}}(C, A)$, then we define $\gamma(f_1, \dots, f_n)$ as

$$\begin{aligned} \gamma(f_1, \dots, f_n) &= C \xrightarrow{\Delta_C} \mathcal{C} \circ C \xrightarrow{pr_n} \mathcal{C}(n) \otimes C^{\otimes n} \\ &\xrightarrow{\gamma \otimes f_1 \otimes \dots \otimes f_n} \mathcal{P}(n) \otimes A^{\otimes n} \xrightarrow{\mu_A} A. \end{aligned}$$

The map pr_n is here the projection on the arity n part of the composition product. It is a straightforward consequence of the definition of the convolution operad that this defines an $Hom_{\mathbb{K}}(\mathcal{C}, \mathcal{P})$ algebra structure on the dg vector space $Hom_{\mathbb{K}}(C, A)$. \square

The next step is to show that the set of morphisms from the L_∞ -operad to an operad \mathcal{Q} is isomorphic to the set of Maurer-Cartan elements in \mathcal{Q} .

The main theorem of this section is the following.

Theorem 7.1. Let $\tau : \mathcal{C} \rightarrow \mathcal{P}$ be an operadic twisting morphism between a dg cooperad \mathcal{C} and a dg operad \mathcal{P} . Let C be a \mathcal{C} -coalgebra and A a \mathcal{P} -algebra, then the L_∞ structure on $Hom_{\mathbb{K}}(C, A)$ described above is natural in both C and A and has the following properties.

1. The Maurer-Cartan elements with respect to this L_∞ structure are the twisting morphisms relative to τ in $Hom_{\mathbb{K}}(\mathcal{C}, A)$.
2. Let \hat{f} and \hat{g} be two Maurer-Cartan elements in $Hom_{\mathbb{K}}(C, A)$ and let $f : \Omega_\tau C \rightarrow A$ and $g : \Omega_\tau C \rightarrow A$ be the corresponding \mathcal{P} algebra maps, then f and g are homotopic in the model category of \mathcal{P} -algebras if and only if the Maurer-Cartan elements \hat{f} and \hat{g} are gauge equivalent.

Proof. According to [15] the set of twisting morphisms is given by

$$\partial(\phi) - \star_\tau(\phi) = 0,$$

where \star_τ is the operator given by

$$\star_\tau(\phi) : C \xrightarrow{\Delta_C} \mathcal{C} \circ C \xrightarrow{\tau \circ \phi} \mathcal{P} \circ A \xrightarrow{\mu_A} A.$$

So we have to show that the Maurer-Cartan equation from [15] is the same as the Maurer-Cartan equation in the L_∞ -algebra $\text{Hom}_{\mathbb{K}}(C, A)$. Therefore we will first make the operations l_n in the L_∞ -structure explicit. This is done in a similar way as for the Lie algebra case as in section 11.1.2 of [15], the operation $l_n(x_1, \dots, x_n)$ is defined as the image of μ_n , the arity n element of COCOM , under the twisting morphism τ . When worked out explicitly it is given by the composite

$$\begin{aligned} l_n(x_1, \dots, x_n) : C &\xrightarrow{\Delta_C} \mathcal{C}(C) \twoheadrightarrow (\mathcal{C}(n) \otimes C^{\otimes n})^{\Sigma_n} \rightarrow \mathcal{C}(n) \otimes C^{\otimes C} \\ &\xrightarrow{\sum_{\sigma \in \Sigma_n} (-1)^\epsilon \tau \otimes x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}} \mathcal{P}(n) \otimes A^{\otimes n} \rightarrow A. \end{aligned}$$

Where the sign ϵ is coming from the Koszul sign rule. From this it follows that $\star_\tau(\phi)$ is equal to $\sum_{n \geq 1} \frac{1}{n!} l_n(\phi, \dots, \phi)$, therefore the twisting morphism Maurer-Cartan equation is equal to the Maurer-Cartan equation coming from the L_∞ -structure.

To prove the second part of the theorem we first recall from [18], that a cylinder object for an algebra A is given by $A \otimes \Omega_1$, where Ω_1 is the commutative algebra of polynomial de Rham forms on the interval. So two morphisms $f, g : A \rightarrow B$ of \mathcal{P} algebras are homotopic if there exists a map $H : A \rightarrow B \otimes \Omega_1$ such that $H|_{t=0} = f$ and $H|_{t=1} = g$. Since the \mathcal{P} -algebra A is of the form $\Omega_\tau(C)$ the morphism $H : \Omega_\tau C \rightarrow A \otimes \Omega_1$ is determined by its restriction to the generators C , i.e. by a Maurer-Cartan element \tilde{H} in the convolution algebra $\text{Hom}_{\mathbb{K}}(C, A \otimes \Omega_1)$. The restriction of \tilde{H} to $t = 0$ is now the Maurer-Cartan element defining f and the restriction to $t = 1$ is the Maurer-Cartan element g . Therefore the maps f and g are homotopic if and only if the corresponding Maurer-Cartan elements are gauge equivalent.

To show that the L_∞ -structure is natural in A , we observe that a morphism $f : A \rightarrow B$ of \mathcal{P} -algebras induces a map between convolution algebras $\tilde{f} : \text{Hom}_{\mathbb{K}}(C, A) \rightarrow \text{Hom}_{\mathbb{K}}(C, B)$ by composition with f . Since f is an algebra morphism it commutes with the \mathcal{P} multiplication maps $\mu_A : \mathcal{P} \circ A \rightarrow A$ and $\mu_B : \mathcal{P} \circ B \rightarrow B$. Therefore it induces a morphism between the corresponding L_∞ -algebras, which proves the naturality in A , the naturality in C is shown in an analogous manner. \square

Several versions of this theorem are already known in the literature, in [15] they prove a similar result for binary quadratic operads and cooperads in Proposition 11.1.1 and Corollary 11.1.2 and in [8] they show that there exists an L_∞ -structure on the convolution algebra. When $\mathcal{C} = B_{op}\mathcal{P}$. This theorem generalizes both results and gives a shorter and more conceptual prove than in [8].

8 Example: The classical Hopf invariant

In this section we will describe a chain level version of the classical Hopf invariant and explain how we will generalize it to an algebraic setting. This section should be seen as a motivation for our generalizations in the next sections. Most of this section comes from Example 1.7 in Section 1 of [17].

The classical Hopf invariant is an invariant of maps $f : S^3 \rightarrow S^2$ and is constructed using the associative bar construction. The goal is to construct a pairing $\eta : H_*(S^3) \times H_*(B_\tau C^*(S^2)) \rightarrow \mathbb{Z}$, which is an invariant of the homotopy classes of maps. To do this let $C^*(S^2)$ be the singular cochains on S^2 and let $\omega \in C^2(S^2)$ be a cocycle that represents the generator of the cohomology of S^2 , such that $\omega^2 = 0$. Since the singular cochains are an associative algebra we can take the associative bar construction $B_\tau C^*(S^2)$ with respect to the twisting morphism $\tau : Ass_{-1}^\vee \rightarrow Ass_0$ between the coassociative operad with a coproduct of degree -1 and the associative operad with a product of degree 0 . Since ω is a cocycle and $\omega^2 = 0$, the element $\omega \otimes \omega$ is also a cycle in the bar construction.

The next step is to pull the form $\omega \otimes \omega$ back to $f^*\omega \otimes f^*\omega \in B_\tau C^*(S^3)$. The cocycle $f^*\omega$ is exact since $H^2(S^3) = 0$, therefore there exists a form $d^{-1}f^*\omega \in C^1(S^3)$. It is straightforward to show that $f^*\omega \otimes f^*\omega$ is homologous to $d^{-1}f^*\omega \cup f^*\omega$, where \cup is the chain level version of the cup product on $C^*(S^3)$.

This version of the classical Hopf invariant of a map $f : S^3 \rightarrow S^2$ is now defined as

$$\int_\alpha (d^{-1}f^*\omega \cup f^*\omega) \in \mathbb{Z}.$$

Which is the evaluation of the 3-form $d^{-1}f^*\omega \cup f^*\omega$ on the fundamental class α of S^3 . It can be shown that this construction is independent of choices and defines an invariant of the map f , which is equal to Hopf's classical definition of the Hopf invariant.

So what we have done in this example is specifying a pairing

$$\eta : H_*(S^3) \otimes H^*(B_\tau C^*(S^2)) \rightarrow \mathbb{Z}.$$

This is equivalent to specifying a linear map $H_*(S^3) \rightarrow H_*(\Omega_\tau C_*(S^2))$. The goal of our generalization of the Hopf invariant is therefore to associate to each map of spaces $f : X \rightarrow Y_{\mathbb{Q}}$ a linear map $mc_\infty : \tilde{H}_*(X) \rightarrow \pi_*(Y_{\mathbb{Q}})$ which is an invariant of the homotopy class of the map.

9 Algebraic Hopf invariants

Let C and D be coalgebras over a cooperad \mathcal{C} and let $\iota : \mathcal{C} \rightarrow \Omega_{op}\mathcal{C}$ be the canonical twisting morphism from \mathcal{C} to its cobar construction. The goal of this section is to construct a map $mc_\infty : Hom_{\mathcal{C}\text{-}coalg}(C, D) \rightarrow \mathcal{MC}(H_*(C), H_*(\Omega_\iota D))$, from the set of coalgebra maps between C and D to a certain moduli space of Maurer-Cartan elements associated to C and D . The map mc_∞ has the property that it is a complete invariant of homotopy classes of maps, i.e. two maps f and g are homotopic if and only if $mc_\infty(f) = mc_\infty(g)$.

The map mc_∞ will be constructed in two steps. The first step is to define a map

$$mc : Hom_{\mathcal{C}\text{-}coalg}(C, D) \rightarrow Hom_{\mathbb{K}}(H_*(C), H_*(\Omega_\iota D)),$$

which assigns to each coalgebra map a Maurer-Cartan element in the convolution algebra between the homology of C and the homology of the cobar construction of D . The cobar construction here is taken with respect to the canonical twisting morphism $\iota : \mathcal{C} \rightarrow \Omega_{op}\mathcal{C}$. The map mc is not homotopy invariant yet, to make it homotopy invariant we have to pass to the moduli space of Maurer-Cartan elements. The second step is

therefore to compose this map with the projection onto the moduli space of Maurer-Cartan elements.

Remark 9.1. To talk about gauge equivalence on the space of Maurer-Cartan elements in $Hom_{\mathbb{K}}(H_*(C), H_*(\Omega_l D))$ it is necessary to have an L_∞ -structure, we will define this L_∞ -structure in Section 11.

The map $mc : Hom_{\mathcal{C}-coalg}(C, D) \rightarrow Hom_{\mathbb{K}}(H_*(C), H_*(\Omega_l D))$ is not canonical and to construct it we first need to make a couple of choices. First we pick an ∞ -quasi-isomorphism $i : H_*(C) \rightarrow C$, where we assume that $H_*(C)$ has a \mathcal{C}_∞ structure coming from the homotopy transfer theorem. Then we pick a strict morphism of $\Omega_{op}\mathcal{C}$ -algebras $p : \Omega_l D \rightarrow H_*(\Omega_l D)$, where we again assume that $H_*(\Omega_l D)$ has a transferred $\Omega_{op}\mathcal{C}$ -structure coming from the homotopy transfer theorem. The morphism p can be chosen in such a way that it is a strict morphism of $\Omega_l\mathcal{C}$ -algebras, and not just an ∞ -morphism, because of the following lemma.

Lemma 9.1. Let $\Omega_l D$ be the cobar construction on D , then $\Omega_l D$ is a quasi-free cofibrant $\Omega_{op}\mathcal{C}$ -algebra and there exists a strict morphism $p : \Omega_l D \rightarrow H_*(\Omega_l D)$ of $\Omega_{op}\mathcal{C}$ algebras such that p is a quasi-isomorphism.

Proof. The algebra $\Omega_l D$ is quasi-free by construction and is cofibrant because of Proposition 5.1. The morphism p is constructed as follows. First we pick an ∞ -quasi-isomorphism $q : \Omega_l D \rightarrow H_*(\Omega_l D)$, from $\Omega_l D$ to $H_*(\Omega_l D)$ with transferred structure coming from the homotopy transfer theorem. According to Theorem 11.4.14 in [15] we can rectify this ∞ -morphism into a zig-zag of strict $\Omega_{op}\mathcal{C}$ algebra morphisms. This is done by using the cobar-bar resolution as follows,

$$\Omega_l D \xleftarrow{\epsilon_{\Omega_l D}} \Omega_l B_l \Omega_l D \xrightarrow{\Omega_l B_l q} \Omega_l B_l H_*(\Omega_l D) \xrightarrow{\epsilon_{H_*(\Omega_l D)}} H_*(\Omega_l D).$$

Where $\epsilon_{\Omega_l D}$ and $\epsilon_{H_*(\Omega_l D)}$ are the counits of the bar cobar adjunction and $\Omega_l B_l p$ is the cobar bar construction applied to the ∞ -quasi-isomorphism p .

Since the object $\Omega_l D$ is cofibrant we can find a homotopy inverse to the quasi-isomorphism $\epsilon_{\Omega_l D}$, which is an inverse on the level of homology. This map is in general not unique, but is unique up to homotopy of $\Omega_{op}\mathcal{C}$ algebras, denote a choice of such a map by $j : \Omega_l D \rightarrow \Omega_l B_l \Omega_l D$.

We can now compose all these maps to obtain a map from $\Omega_l D$ to $H_*(\Omega_l D)$, which is given by

$$\Omega_l D \xrightarrow{j} \Omega_l B_l \Omega_l D \xrightarrow{\Omega_l B_l q} \Omega_l B_l H_*(\Omega_l D) \xrightarrow{\epsilon_{H_*(\Omega_l D)}} H_*(\Omega_l D).$$

Since j , $\Omega_l B_l q$ and $\epsilon_{H_*(\Omega_l D)}$ are strict quasi-isomorphisms of $\Omega_l\mathcal{C}$ algebras, their composite $\epsilon_{H_*(\Omega_l D)} \circ \Omega_l B_l q \circ j$ is also a strict quasi-isomorphism of $\Omega_{op}\mathcal{C}$ algebras, which proves the lemma. \square

Now that we have fixed the maps i and p we define the map as follows

$$mc : Hom_{\mathcal{C}-coalg}(C, D) \rightarrow Hom_{\mathbb{K}}(H_*(C), H_*(\Omega_l D))$$

$$mc(f) = p \circ \Omega_l f \circ \Omega_l i.$$

So we first take the cobar construction of f and then precompose it with $\Omega_l i$ and compose it with p . This map is not yet homotopy invariant but homotopic maps will have gauge equivalent values. Therefore we define $mc_\infty : Hom_{\mathcal{C}-coalg}(C, D) \rightarrow \mathcal{MC}(H_*(C), H_*(\Omega_l D))$ as the map which sends f to the equivalence class of $mc(f)$ in the moduli space of Maurer-Cartan elements.

Definition 9.1. The algebraic Hopf invariant $mc_\infty(f)$ of a map $f : C \rightarrow D$ is defined as the image of the map $mc_\infty : Hom_{C-coalg}(C, D) \rightarrow MC(H_*(C), H_*(\Omega_i D))$, where mc_∞ is the map that sends f to the equivalence class of $mc(f)$ in the moduli space of Maurer-Cartan elements.

We will spend the rest of this section to show that mc_∞ is a well defined invariant of the set of homotopy classes of maps. In the next section we will show that it is a complete invariant as well.

Proposition 9.1. For every choice of maps $i : H_*(C) \rightarrow C$ and $p : \Omega_i D \rightarrow H_*(\Omega_i D)$, the algebraic Hopf invariant is an invariant of the homotopy class of the map f .

Before we prove the proposition we will first prove the following lemma. This lemma should be well known, but since we could not find a reference we decided to include it for the sake of completeness.

Lemma 9.2. Let $f, g : C \rightarrow D$ be left homotopic maps in a model category \mathbf{C} , assume that the objects C and D are cofibrant and let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a left Quillen functor from \mathbf{C} to a model category \mathbf{D} . Then F preserves left homotopies, i.e. $F(f)$ is left homotopic to $F(g)$.

Proof. Let $Cyl(C)$ be a good cylinder object for C , which gives a homotopy between f and g . This is an object $Cyl(C)$, such that the first map is a cofibration and the second map a weak equivalence $C \sqcup C \hookrightarrow Cyl(C) \rightarrow C$, where $C \sqcup C$ is the coproduct. Since f and g are homotopic this means that there is a map $H : Cyl(C) \rightarrow D$ such that H restricted to the first factor is f and restricted to the second factor is g . So what we would like to prove is that $F(Cyl(C))$ is a cylinder object for $F(C)$, because then the map $F(H) : F(Cyl(C)) \rightarrow F(D)$ is a homotopy between $F(f)$ and $F(g)$.

To prove that $F(Cyl(C))$ is a cylinder object for $F(C)$ we first observe that since F is a left Quillen functor it preserves weak equivalences between cofibrant objects and since it is a left adjoint it preserves coproducts. Since we assumed that C is cofibrant this implies that $C \sqcup C$ and $Cyl(C)$ are also cofibrant. So if we apply the functor F then we get $F(C \sqcup C) \hookrightarrow F(Cyl(C)) \rightarrow F(C)$ which is equal to $F(C) \sqcup F(C) \hookrightarrow F(Cyl(C)) \rightarrow F(C)$. Therefore $F(Cyl(C))$ is a cylinder object for $F(C)$ and $F(H)$ a homotopy between $F(f)$ and $F(g)$. \square

The homotopy invariance of the Hopf invariants follows from this lemma if we apply it to the cobar construction.

Proof of Proposition 9.1. The homotopy invariance of the algebraic Hopf invariants follows from Lemma 9.2 and Theorem 7.1. More precisely if $f : C \rightarrow D$ and $g : C \rightarrow D$ are homotopic maps then $\Omega_i f$ and $\Omega_i g$ are homotopic because of Lemma 9.2. The lemma applies since Ω_i is a left Quillen functor and every coalgebra is cofibrant. Since $\Omega_i f$ and $\Omega_i g$ are homotopic, so are $p \circ \Omega_i f \circ i$ and $p \circ \Omega_i g \circ i$. Then because of Theorem 7.1 the maps $p \circ \Omega_i f \circ i$ and $p \circ \Omega_i g \circ i$ are homotopic if and only if the corresponding Maurer-Cartan elements are gauge equivalent. Therefore the algebraic Hopf invariants are an invariant of the homotopy class of the map f . \square

Remark 9.2. The choice of the maps i and p is already visible in the classical case, since we need to pick an orientation for the fundamental class of S^{4n-1} . If we do not pick an orientation the classical Hopf invariant is only well defined up to a sign. This choice of an orientation is exactly

what the map i does in our construction, it fixes a set of representatives for the homology for which we compute the corresponding Maurer-Cartan element.

Since we need to make a choice for i and p we will assume for simplicity in the rest of this paper that the maps i and p are fixed and we will omit them from the notation.

10 Completeness of the algebraic Hopf invariants

In this section we will prove the main theorem about the algebraic Hopf invariants, which is that they form a complete invariant of the set of homotopy classes of maps $[C, D]$ between two \mathcal{C} -coalgebras C and D .

Theorem 10.1. Let $f : C \rightarrow D$ and $g : C \rightarrow D$ be two maps of \mathcal{C} coalgebras, then f and g are homotopic in the model category of \mathcal{C} -coalgebras if and only if they have the same algebraic Hopf invariant.

Proof. To prove the theorem we want to show that the algebraic Hopf invariant map mc_∞ induces a bijection between the set of homotopy classes of maps $[C, D]$ and the moduli space of Maurer-Cartan elements $\mathcal{MC}(H_*(C), H_*(\Omega_i D))$. Since two maps are homotopic if and only if they belong to the same homotopy class of maps, this bijection implies that two maps are homotopic if and only if they have the same algebraic Hopf invariant.

To show that there is a bijection between $[C, D]$ and $\mathcal{MC}(H_*(C), H_*(\Omega_i D))$ we will show that we have the following sequence of bijections.

$$\begin{aligned} [C, D] &\cong [\Omega_i C, \Omega_i D] \cong [\Omega_i C, H_*(\Omega_i D)] \cong \\ &[\Omega_i H_*(C), H_*(\Omega_i D)] \cong \mathcal{MC}(H_*(C), H_*(\Omega_i D)). \end{aligned}$$

The first bijection between $[C, D]$ and $[\Omega_i C, \Omega_i D]$ follows from the fact that Ω_i is the left Quillen functor in a Quillen equivalence and therefore induces a bijection on the level of homotopy categories.

The second bijection between $[\Omega_i C, \Omega_i D]$ and $[\Omega_i C, H_*(\Omega_i D)]$ is given by the map induced by composition with the map $p : \Omega_i D \rightarrow H_*(\Omega_i D)$ from Lemma 9.1. Since the map p is a surjective quasi-isomorphism it is an acyclic fibration, Lemma 5.1 then implies that the second bijection is indeed a bijection.

To show that we have the third bijection between $[\Omega_i C, H_*(\Omega_i D)]$ and $[\Omega_i H_*(C), H_*(\Omega_i D)]$ we want to use Lemma 5.2. We apply this lemma to the functor $[-, H_*(\Omega_i D)] : \Omega_{op}\mathcal{C}\text{-algebras} \rightarrow \text{Sets}$ which sends an algebra A to the set of homotopy classes of maps between A and $H_*(\Omega_i D)$. The category of sets is here equipped with the trivial model structure in which the weak equivalences are given by bijections. This functor carries acyclic cofibrations to weak equivalences because of a dual version of Lemma 5.1, therefore K. Brown's Lemma applies. Since both $\Omega_i C$ and $\Omega_i H_*(C)$ are cofibrant and the map $\Omega_i i : \Omega_i H_*(C) \rightarrow \Omega_i C$ is a weak equivalence, the map $[\Omega_i C, H_*(\Omega_i D)] \rightarrow [\Omega_i H_*(C), H_*(\Omega_i D)]$ is a weak equivalence in the category of sets and therefore a bijection.

The last bijection between $[\Omega_i H_*(C), H_*(\Omega_i D)]$ and $\mathcal{MC}(H_*(C), H_*(\Omega_i D))$ follows from Theorem 7.1 and the fact that the set of twisting morphisms is represented by the cobar construction. More

precisely, the set of algebra morphisms $Hom_{\Omega_{op}C-alg}(\Omega_\iota H_*(C), H_*(\Omega_\iota D))$ is in bijection with the Maurer-Cartan elements in the convolution L_∞ -algebra $Hom_{\mathbb{K}}(H_*(C), H_*(\Omega_\iota D))$. Because of Theorem 7.1 the homotopy equivalence relation on $Hom_{\Omega_{op}C-alg}(\Omega_\iota H_*(C), H_*(\Omega_\iota D))$ is equivalent to the gauge equivalence relation on $Hom_{\mathbb{K}}(H_*(C), H_*(\Omega_\iota D))$. Therefore there is a bijection between $[\Omega_\iota H_*(C), H_*(\Omega_\iota D)]$ and $MC((H_*(C), H_*(\Omega_\iota D)))$.

Putting all these bijections together we get a bijection between $MC(H_*(C), H_*(\Omega_\iota D))$ and $[C, D]$ which proves the theorem. \square

11 Models for mapping spaces

Let X and Y be spaces and A be a non-unital CDGA model for X and L an L_∞ model for Y , in [4] Berglund proves that an L_∞ -model for the based mapping space $Map_*(X, Y)$ is given by $A \otimes L$. Although this construction works in many interesting cases it has the obvious disadvantage that it assumes that the source A has to be a CDGA instead of a more general C_∞ -algebra. In this section we extend his result by constructing a rational model for the mapping space from a C_∞ -coalgebra and an L_∞ -algebra. As a corollary we give an alternative proof for a theorem by Buijs and Gutiérrez from [6] which states that $\tilde{H}^*(X) \otimes \pi_*(Y)$ can be equipped with an L_∞ -structure such that it becomes a rational model for the mapping space. From now on it will be necessary to have some restrictions on to the L_∞ -algebras we are considering. This is done in the following convention.

Convention 11.1. From now on we will assume that all L_∞ -algebras are degree-wise nilpotent (see [4] for a definition and discussion about degree-wise nilpotence). Under some mild assumptions on X and Y , the models for the mapping space $Map_*(X, Y)$ will always be degree-wise nilpotent. An example of such assumptions would be to assume that X is a finite 1-reduced CW-complex and Y a simply connected rational space of finite \mathbb{Q} -type. Therefore we will from now on tacitly assume that all the L_∞ -algebras we encounter are degree-wise nilpotent.

To construct our model for the mapping space we begin by recalling the main theorems from [4], the following theorems are a version of Theorem 1.5 in [4] and Corollary 1.2 in [4].

Theorem 11.1. Let X be a finite simply connected CW-complex and let Y be a nilpotent space of finite \mathbb{Q} -type and $Y_{\mathbb{Q}}$ its \mathbb{Q} -localization. If A is a CDGA model of finite \mathbb{Q} -type for X and L an L_∞ -model for $Y_{\mathbb{Q}}$ then there is a homotopy equivalence of simplicial sets

$$Map_*(X, Y_{\mathbb{Q}}) \simeq MC_\bullet(A \hat{\otimes} L).$$

The symbol $\hat{\otimes}$ is the complete tensor product, for more details see [4].

Theorem 11.2. An L_∞ -morphism between degree-wise nilpotent L_∞ -algebras $f : L \rightarrow M$ induces an equivalence between the Maurer-Cartan simplicial sets $MC_\bullet(L) \rightarrow MC_\bullet(M)$ if and only if the following two conditions are satisfied:

1. The map $MC(L) \rightarrow MC(M)$ on moduli spaces of Maurer-Cartan elements is a bijection.
2. The L_∞ -morphism $f^\tau : L^\tau \rightarrow (M)^{f^*(\tau)}$ induces an isomorphism in homology in non negative degrees for every Maurer-Cartan element τ in L .

We will combine these results and the L_∞ -structure from section 7 to construct a model for the mapping space as follows. Let C be a finite dimensional C_∞ -coalgebra model for a simply connected finite CW complex X and let L be an L_∞ -model of finite type for a rational space of finite \mathbb{Q} -type $Y_\mathbb{Q}$. We would like to apply Theorem 7.1 to the convolution algebra $\text{Hom}_\mathbb{K}(C, A)$, but to apply this theorem we first need a twisting morphism $\tau : C_\infty \rightarrow L_\infty$. In Theorem 6.5.10 in [15] it is shown that the set of twisting morphisms in $\text{Hom}_\mathbb{K}(C_\infty, L_\infty)$ is represented by the set of cooperad maps $\text{Hom}_{\text{coOp}}(C_\infty, B_{op}L_\infty)$. Therefore to construct the twisting morphism τ , we pick a quasi-isomorphism $\phi : C_\infty \rightarrow B_{op}\Omega_{op}\text{COCOM}$ from the bar-cobar resolution of the cocommutative cooperad COCOM to C_∞ . Since both C_∞ and $B_{op}\Omega_{op}\text{COCOM}$ are both fibrant and cofibrant this is always possible. The twisting morphism τ is then defined as the composition of ϕ with the universal twisting morphism $\pi : B_{op}L_\infty \rightarrow L_\infty$ given by the projection onto L_∞ , i.e. $\tau = \pi \circ \phi$. The morphism ϕ also allows us to view every C_∞ -coalgebra as an $B_{op}\Omega_{op}\text{COCOM}$ -algebra, since this does not change the underlying chain complex and therefore the homology, it will be convenient to view every C_∞ -coalgebra as a $B_{op}\Omega_{op}\text{COCOM}$ -coalgebra. Therefore we will from now view every C_∞ -coalgebra as an $B_{op}\Omega_{op}\text{COCOM}$ -coalgebra, unless stated otherwise, this has the advantage that we can work with the twisting morphism π which is easier than τ .

Theorem 11.3. In the situation described above the convolution L_∞ -algebra $\text{Hom}_\mathbb{K}(C, L)$ is a model for the mapping space $\text{Map}_*(X, Y_\mathbb{Q})$, i.e. there is a homotopy equivalence between

$$\text{Map}_*(X, Y_\mathbb{Q}) \simeq MC_\bullet(\text{Hom}_\mathbb{K}(C, L)).$$

Proof. To prove the theorem, we want to use Berglund's theorem to show that $\text{Hom}_\mathbb{K}(C, L)$ is a model for the mapping space, unfortunately we can not apply this theorem directly since C is not a cocommutative coalgebra, therefore we first need to replace C by a cocommutative coalgebra. This is done by applying the bar-cobar resolution to C , first we apply the cobar construction relative to the operadic twisting morphism $\pi : B_{op}\Omega_{op}\text{COCOM} \rightarrow \Omega_{op}\text{COCOM}$ and then we use the bar construction relative to $\iota : \text{COCOM} \rightarrow \Omega_{op}\text{COCOM}$. Because of Theorem 11.4.4 in [15], the result $B_\iota\Omega_\pi C$ is cocommutative coalgebra quasi-isomorphic to C when viewed as a $B_{op}\Omega_{op}\text{COCOM}$ -algebra and is therefore also a model for the space X . The L_∞ -algebra $\text{Hom}_\mathbb{K}(B_\iota\Omega_\pi C, L)$ is then by Theorem 11.1 a model for $\text{Map}_*(X, Y_\mathbb{Q})$.

To prove the theorem we first note that there is a canonical morphism of $B_{op}\Omega_{op}\text{COCOM}$ -coalgebras $\epsilon : C \rightarrow B_\iota\Omega_\pi C$, which is defined in Proposition 11.4.3 and Theorem 11.4.4 in [15]. The morphism ϵ induces a quasi-isomorphism of L_∞ -algebras $\tilde{\epsilon} : \text{Hom}_\mathbb{K}(B_\iota\Omega_\pi C, L) \rightarrow \text{Hom}_\mathbb{K}(C, L)$ given by precomposition by ϵ . Since the morphism ϵ is a quasi-isomorphism the morphism $\tilde{\epsilon}$ is also a quasi-isomorphism. To prove the theorem we need to show that this morphism satisfies the conditions from Theorem 11.2.

To prove that the map $\tilde{\epsilon}$ induces a bijection between the moduli spaces of Maurer-Cartan elements we will again use the Ken Brown Lemma (see Lemma 5.2). We want to apply the lemma to the functor $[-, L] : B_{op}\Omega_{op}\text{COCOM-coalg} \rightarrow \text{Sets}$, which sends a $B_{op}\Omega_{op}\text{COCOM}$ -coalgebra C to the set of homotopy classes of maps between $\Omega_\pi C$ and L . Because of Theorem 7.1, there is a bijection between $\mathcal{MC}(C, L)$ and $[\Omega_\pi C, L]$. So if we can show that ϵ induces a bijection between $[\Omega_{op} C, L]$ and $[\Omega_\pi B_\pi \Omega_\iota C, L]$, then this implies that there is a bijection between the moduli spaces of

Maurer-Cartan elements $\mathcal{MC}(C, L)$ and $\mathcal{MC}(B_\pi \Omega_\iota C, L)$. To show this we use Lemma 5.2. The lemma applies because by Proposition 5.1 every coalgebra is cofibrant and because of the dual of Lemma 5.1. Lemma 5.1 implies that every acyclic cofibration between $B_{op} \Omega_{op} \mathcal{COM}$ -coalgebras induces a bijection between the corresponding sets of homotopy classes of maps and therefore induces a bijection between $[\Omega_{op} C, L]$ and $[\Omega_\pi B_\pi \Omega_\iota C, L]$. So Lemma 5.2 applies and since the map $\tilde{\epsilon}$ is a weak equivalence, it therefore induces a bijection between $[\Omega_{op} C, L]$ and $[\Omega_\pi B_\pi \Omega_\iota C, L]$.

To prove that the map $\tilde{\epsilon}^\kappa : Hom_{\mathbb{K}}(B_\iota \Omega_\pi C, L)^\kappa \rightarrow Hom_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)}$ induces an isomorphism in homology in non-negative degrees for every Maurer-Cartan element κ , we first note that since $\tilde{\epsilon}$ is a quasi-isomorphism, it induces an isomorphism in homology when κ is the zero Maurer-Cartan element. To prove that $\tilde{\epsilon}^\kappa$ is a quasi-isomorphism for non-zero Maurer-Cartan elements κ , we will define two spectral sequences associated to $Hom_{\mathbb{K}}(B_\iota \Omega_\pi C, L)^\kappa$ and $Hom_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)}$ and then apply the Eilenberg-Moore Comparison Theorem (see [19] Theorem 5.5.11).

To define the spectral sequences for $Hom_{\mathbb{K}}(B_\iota \Omega_\pi C, L)^\kappa$ and $Hom_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)}$ we first define the support of a function. Let $f : C \rightarrow L$ be a map, then we define the support of f as $supp(f) = \{c \in C \mid f(c) \neq 0\}$. Denote by $C^{\geq n}$ the subspace of C of all elements of degree less or equal to n . The descending filtrations are then defined as follows.

$$F^p Hom_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)} = \{f \in (Hom_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)}) \mid supp(f) \subseteq C^{\geq p}\}$$

and

$$F^p Hom_{\mathbb{K}}(B_\iota \Omega_\pi C, L)^\kappa = \{f \in Hom_{\mathbb{K}}(B_\iota \Omega_\pi C, L)^\kappa \mid supp(f) \subseteq C^{\geq p}\}.$$

Since we want to use the Eilenberg-Moore Comparison Theorem we should show that these filtrations are stable under the differential, exhaustive and complete. First we will show that the filtrations are stable under the differentials. For simplicity we will only consider the filtration for $Hom_{\mathbb{K}}(B_\iota \Omega_\pi C, L)$, since the situation for $Hom_{\mathbb{K}}(C, L)$ is completely analogous or even simpler.

The differential of $Hom_{\mathbb{K}}(B_\iota \Omega_\pi C, L)$ is given by

$$d^\kappa(f) = d_L \circ f - (-1)^{|f|} f \circ d_{B_\iota \Omega_\pi C} + l_2(f, \kappa) + l_3(f, \kappa, \kappa) + \dots$$

We want to show that if $supp(f) \subseteq C^{\geq p}$ then $supp(d^\kappa(f)) \subseteq C^{\geq p}$, to do this we will analyze each term of the differential and show that it does not lower the support. The term $d_L \circ f$ does not lower the support because d_L does not change anything in the source of the map f . The term $f \circ d_{B_\iota \Omega_\pi C}$ does not lower the support, because if $x \in B_\iota \Omega_\pi C$ is an element of degree p then $f d_{B_\iota \Omega_\pi C}(x) = 0$ since f vanishes on degree $p-1$ elements. To show that the operations $l_n(-, \kappa, \dots, \kappa)$ do not lower the support, we will first recall that the operation l_n is defined by

$$l_n(f_1, \dots, f_n) : C \xrightarrow{\Delta_{B_\iota \Omega_\pi C}} (B_{op} \Omega_{op} \mathcal{C}(n) \otimes C^{\otimes n})_{S_n} \\ \xrightarrow{\iota \otimes f_1 \otimes \dots \otimes f_n} (\Omega_{op} \mathcal{C} \otimes L^{\otimes n})_{S_n} \rightarrow L.$$

The coproduct $\Delta_{B_\iota \Omega_\pi C}(x)$ of an element $x \in B_\iota \Omega_\pi C$ of degree p , will be of the form $\sum c \otimes x_1 \otimes \dots \otimes x_n$ with $c \in B_{op} \Omega_{op} \mathcal{C}$ and $x_i \in B_\iota \Omega_\pi C$. Since $B_\iota \Omega_\pi C$ is 1-reduced, each element x_i will have degree at most $p-1$. Since

we assumed that the morphism f vanishes on elements of degree less than p , $f(x_i) = 0$ for all x_i . The operation l_n therefore preserves the filtration, which proves that the filtration is stable under the differential.

To show that the filtration is exhaustive we notice that it is a descending filtration and that $F_0 \text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L)$ is equal to $\text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L)$, which proves that the filtration is exhaustive. To show that the filtration is complete we need to show that

$$\text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L) = \varprojlim \text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L) / F_p \text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L).$$

The space $\text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L) / F_p \text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L)$ is given by the space of all function whose support is in degree lower than p , which we will denote by $\text{Hom}_{\mathbb{K}}((B_t \Omega_\pi C)^{\leq p-1}, L)$. So we have to prove that $\varprojlim \text{Hom}_{\mathbb{K}}((B_t \Omega_\pi C)^{\leq p-1}, L)$ is equal to $\text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L)$. To do this we will use the fact that the functor $\text{Hom}_{\mathbb{K}}(-, L)$ turns colimits into limits. Therefore $\varprojlim \text{Hom}_{\mathbb{K}}((B_t \Omega_\pi C)^{\leq p-1}, L)$ is isomorphic to $\text{Hom}_{\mathbb{K}}(\text{colim}(B_t \Omega_\pi C)^{\leq p-1}, L)$, since $\text{colim} B_t \Omega_\pi C^{\leq p-1} = B_t \Omega_\pi C$ this implies that $\text{Hom}_{\mathbb{K}}(\text{colim}(B_t \Omega_\pi C)^{\leq p-1}, L)$ is equal to $\text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L)$, which proves that the filtration is complete.

To apply the Eilenberg-Moore Comparison Theorem we now need to show that the map $\tilde{\epsilon}^\kappa$ induces an isomorphism on the E^2 -page. The E^0 pages of the corresponding spectral sequences are then given by

$$E_{p,q}^0 \text{Hom}_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)} = \text{Hom}_{\mathbb{K}}(C_{-p}, L_{p+q})$$

and

$$E_{p,q}^0 \text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L)^\kappa = \text{Hom}_{\mathbb{K}}((B_t \Omega_\pi C)_{-p}, L_{p+q}).$$

The first differential is the part of the differential that does not change the support of a function f at all and is given by d_L . The E^1 -pages therefore become

$$E_{p,q}^1 \text{Hom}_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)} = \text{Hom}_{\mathbb{K}}(C_{-p}, H_{p+q}(L))$$

and

$$E_{p,q}^1 \text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L)^\kappa = \text{Hom}_{\mathbb{K}}((B_t \Omega_\pi C)_{-p}, H_{p+q}(L)).$$

The second differentials in the spectral sequences are given by d_C and $d_{B_t \Omega_\pi C}$. Since d_C lowers the degree of the source by 1, it raises the support of f by one degree. The support of the Maurer-Cartan element κ is of degree at least 2, since C and $B_t \Omega_\pi C$ are both 1 reduced. Therefore taking the bracket l_n with κ will raise the support of f by at least 2, which means that the part of the differential consisting of the operations $l_n(f, \kappa, \dots, \kappa)$ does not have the right target to contribute to the second differential. The second differentials are therefore given by d_C and $d_{B_t \Omega_\pi C}$. The E^2 -pages then become:

$$E_{p,q}^2 \text{Hom}_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)} = \text{Hom}_{\mathbb{K}}(H_{-p}(C), H_{p+q}(L))$$

and

$$E_{p,q}^2 \text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L)^\kappa = \text{Hom}_{\mathbb{K}}(H_{-p}(B_t \Omega_\pi C), H_{p+q}(L)).$$

The map $\tilde{\epsilon}^\kappa$ now induces an isomorphism between the E^2 pages of the spectral sequences. Because the map ϵ is a quasi-isomorphism $H_*(\text{Hom}_{\mathbb{K}}(C, L))$ is isomorphic to $H_*(\text{Hom}_{\mathbb{K}}(B_t \Omega_\pi C, L))$. The Eilenberg-Moore Comparison Theorem now states that since the filtrations of

$Hom_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)}$ and $Hom_{\mathbb{K}}(B_t \Omega_{\pi} C, L)^{\kappa}$ are both complete and exhaustive, the map $H_*(\tilde{\epsilon}^{\kappa}) : H_*(Hom_{\mathbb{K}}(B_t \Omega_{\pi} C, L)^{\kappa}) \rightarrow H_*(Hom_{\mathbb{K}}(C, L)^{\kappa})$ is also an isomorphism.

Since the map $\tilde{\epsilon}^{\kappa}$ induces an isomorphism between the homologies of $Hom_{\mathbb{K}}(C, L)^{\tilde{\epsilon}(\kappa)}$ and $Hom_{\mathbb{K}}(B_t \Omega_{\pi} C, L)^{\kappa}$, it is in particular an isomorphism in non-negative degrees. Therefore Theorem 11.2 applies and shows that there is a homotopy equivalence between $MC_{\bullet}(Hom_{\mathbb{K}}(C, L))$ and $MC_{\bullet}(B_t \Omega_{\pi} C, L)$, combined with Theorem 11.1 this proves that $Hom_{\mathbb{K}}(C, L)$ is a model for $Map_*(X, Y_{\mathbb{Q}})$. \square

Remark 11.1. The coalgebra $B_t \Omega_{\pi} C$ is not necessarily degree-wise nilpotent, but since it is complete Theorem 11.3 still applies.

As a corollary of Theorem 11.3 we find a new proof for Theorem 3.2 in the paper [6] by Buijs and Gutiérrez which states that $Hom_{\mathbb{K}}(\tilde{H}_*(X), \pi_*(Y))$ can be equipped with an L_{∞} -structure such that it becomes a model for $Map_*(X, Y_{\mathbb{Q}})$.

Corollary 11.1. Let X be a finite 1-reduced CW-complex and $Y_{\mathbb{Q}}$ a simply connected rational space of finite \mathbb{Q} -type, then the space $Hom_{\mathbb{K}}(\tilde{H}_*(X), \pi_*(Y))$ can be equipped with an L_{∞} -structure such that

$$MC_{\bullet}(Hom_{\mathbb{K}}(\tilde{H}_*(X), \pi_*(Y))) \simeq Map_*(X, Y_{\mathbb{Q}}).$$

Proof. To prove the corollary we define H as the reduced homology of X with a C_{∞} -coalgebra structure such that H becomes a C_{∞} -model for X and let L be $\pi_*(Y)$ with an L_{∞} -structure such that L is an L_{∞} -model for Y . If we turn H into an $B_{op} \Omega_{op} C O C O M$ model using the morphism ϕ we can form the convolution algebra $Hom_{\mathbb{K}}(H, L)$, by Theorem 11.3 this is a model for $Map_*(X, Y_{\mathbb{Q}})$, which proves the corollary. \square

12 Application of the algebraic Hopf invariants: Rational homotopy theory

In this section we discuss how to apply the algebraic Hopf invariants to rational homotopy theory and connect them to the work of Sinha and Walter in [17]. In [17] Sinha and Walter study spaces of maps from the n -dimensional sphere to a rational target space $Y_{\mathbb{Q}}$. One of their results is that they give a complete invariant of maps from the sphere to another space. In this section we will show how we recover their work by applying the algebraic Hopf invariants to $Map_*(S^n, Y_{\mathbb{Q}})$ and how to generalize this to general mapping spaces.

The starting point is to use the functor from Theorem 4.2 and apply this to the spaces X and Y . In particular two maps $f, g : X \rightarrow Y$ between X and Y are rationally homotopic if and only if $\mathcal{C}\lambda(f) : \mathcal{C}\lambda(X) \rightarrow \mathcal{C}\lambda(Y)$ and $\mathcal{C}\lambda(g) : \mathcal{C}\lambda(X) \rightarrow \mathcal{C}\lambda(Y)$ are homotopic as coalgebra maps.

When we apply this to maps from the sphere S^n to a general target space Y we get a map

$$Map_*(S^n, Y) \rightarrow Hom_{CDGC}(\mathcal{C}\lambda(S^n), \mathcal{C}\lambda(Y)) \xrightarrow{mc_{\infty}} \mathcal{MC}(S^n, Y),$$

from the space of maps between S^n and Y to the moduli space of Maurer-Cartan elements of $Hom_{\mathbb{K}}(\mathcal{C}\lambda(S^n), \Omega_t \mathcal{C}\lambda(Y))$ which for simplicity we will denote by $\mathcal{MC}(S^n, Y)$. By Theorem 10.1 this map is a complete invariant of homotopy classes of maps, i.e. the map $mc_{\infty} : [S^n, Y] \rightarrow \mathcal{MC}(S^n, Y)$

is a bijection. Using this we can give an alternative completely algebraic proof of Theorem 2.10 in [17].

Theorem 12.1. Let Y be a 1-reduced space, there is an isomorphism of groups between $mc_\infty : \pi_n(Y) \rightarrow H_n(\Omega_i \mathcal{C}\lambda(Y))$.

Proof. To prove the theorem we want to use the completeness of the algebraic Hopf invariants, to do this we will first need to compute the moduli space of Maurer-Cartan elements. The moduli space of Maurer-Cartan elements is independent of the models chosen for S^n and Y , since the set of homotopy classes of maps between S^n and Y is independent of the models chosen (as long as the models are of good enough quality, i.e. S^n cofibrant and Y fibrant). Note the Hopf invariant does depend on the models chosen and in particular on the maps i and p .

To compute the moduli space of Maurer-Cartan elements we pick for S^n the CDGC model given by a 1-dimensional vector space in degree n with the trivial comultiplicative structure, by abuse of notation we will denote this by S^n . A CDGC model for Y will also by abuse of notation denoted by Y , a Lie model for Y is then given by $\Omega_i Y$. The convolution L_∞ algebra between S^n and $\Omega_i Y$ is now given by $Hom_{\mathbb{K}}(S^n, \Omega_i Y)$, since S^n is 1-dimensional this is isomorphic as dg vector spaces to the n -fold desuspension of Y . Since $\Delta_{S^n} = 0$ all the brackets l_n on the convolution algebra $Hom_{\mathbb{K}}(S^n, \Omega_i Y)$ are zero except for l_1 which is the desuspended differential of $\Omega_i Y$.

The Maurer-Cartan equation for an element $x \in Hom_{\mathbb{K}}(S^n, \Omega_i Y)$ is therefore given by $d(x) = 0$ and the space of Maurer-Cartan elements can be identified with the space of cycles in Y_n . The gauge equivalence relation on the set of Maurer-Cartan elements is given by the relation of homology, i.e. two cycles are gauge equivalent if and only if they are homologous. So $\mathcal{MC}(S^n, Y)$ is isomorphic to $H_n(\Omega_i Y)$.

Because the Hopf invariants are a complete invariant there is an explicit bijection of sets from $[S^n, Y]$ to $H_n(\Omega_i Y)$ given by $mc_\infty : [S^n, Y] \rightarrow H_n(\Omega_i Y)$. So what is left to show is that this is an isomorphism of groups. The group structure on $\pi_n(Y)$ is coming from the pinch map $S^n \rightarrow S^n \vee S^n$ which algebraically is modeled by the diagonal map $\delta : \mathbb{Q} \rightarrow \mathbb{Q} \oplus \mathbb{Q}$ $\delta(x) = x \oplus x$. The group structure is then given by $f * g(x) = f(x) + g(x)$ for two maps $f, g : S^n \rightarrow Y$, this is exactly the same as the group structure on the moduli space of Maurer-Cartan elements $\mathcal{MC}(S^n, Y)$, which is given by addition of the elements. \square

The algebraic Hopf invariants have the advantage that they generalize the work of Sinha and Walter to more general spaces. In particular they are a complete invariant of rational homotopy classes of maps.

Theorem 12.2. Let $f, g : X \rightarrow Y$ be maps from a finite CW-complex X to a space Y , then f and g are rationally homotopic if and only if they have the same algebraic Hopf invariant, i.e. $mc_\infty(f) = mc_\infty(g)$.

Proof. The Theorem follows immediately from the completeness Theorem 10.1. \square

Although the theorem gives a complete invariant of homotopy classes of maps, it might be hard to compute the actual values of this invariant. In future work we are planning to construct some easier computable invariants from the algebraic Hopf invariants.

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